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The Further Study of a Certain Nonlinear Integro-differential Equation

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In this paper the writers give a global existence and uniqueness theorem for a nonlinear integro-differential equation which occurs in the statistical theory of turbulent diffusion. The theorem is more general than that presented by S. H. Chang and J. T. Day (*J. Comput. Phys.* **26** (1978), 162). The problems of the existence and uniqueness of a solution to the equation are solved completely. Implicit Runge–Kutta methods with m stages and optimal order $p=2m$ for the approximate solution of the equation are introduced. Computational examples are also considered.

1. INTRODUCTION

Consider the nonlinear integro-differential equation

$$u'(t)+a(t)\,u(t)+\int_0^t k(t,s)\,u(t-s)\,u(s)\,ds=f(t), \qquad 0\leqslant t\leqslant T, u(0)=c, \tag{1.1}$$

where the functions $a(t)$, $f(t)$ and $k(t,s)$ are continuous for $0\leqslant s\leqslant t\leqslant T$, and c is a constant. Equations of this type occur as model equations for describing turbulent diffusion (see Velikson [2] and Monin and Yaglom [3]). In [1] an existence and uniqueness theorem for Eq. (1.1) and a numerical algorithm are given, but the theorem and the numerical method are merely valid under the following conditions:

(i) $a(t)\geqslant 0$ for all $0\leqslant t\leqslant T$, and

(ii) $|c|+\int_0^T |f(t)|\,dt\leqslant\frac{1}{2}$, $\int_0^T\int_0^T |k(r,s)|\,ds\,dr<\frac{1}{2}$.

In Section 2 of this paper we remove the above restrictions and prove a global existence and uniqueness theorem for Eq. (1.1) using Schauder's fixed point theorem and some inequalities. It is also proved that the smoothness properties of the solution of (1.1) depend merely on those of the functions $a(t)$, $f(t)$ and $k(t,s)$.

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In Section 3, implicit Runge–Kutta methods for finding the numerical solution of (1.1) are introduced. These methods can be viewed as fully discretized collocation methods in certain polynomial spline spaces (see also [5]). We could compute the desired numerical solution through solving a system of nonlinear equations and some systems of linear equations. These methods are high-order numerical methods, and global discretization error estimates for the methods are obtained. In Section 4, the formulas of a 4th-order implicit Runge–Kutta method are given. Several computational examples are considered. It is found that the method we used has two major advantages: stability and accuracy.

2. Main Theorems

Equation (1.1) can be transformed to an equivalent integro-differential equation. Let

$$A(t) = \int_0^t a(s)\, ds.$$

Multiplying Eq. (1.1) by $e^{A(t)}$, we obtain

$$(e^{A(t)}\, u(t))' + \int_0^t e^{A(t)}\, k(t, s)\, u(t-s)\, u(s)\, ds = f(t)\, e^{A(t)}. \tag{2.1}$$

Then, we have

$$U'(t) + \int_0^t K(t, s)\, U(t-s)\, U(s)\, ds = F(t), \qquad 0 \leqslant t \leqslant T,\ U(0) = c, \qquad (2.2)'$$

where

$$U(t) = u(t)\, e^{A(t)}, \qquad F(t) = f(t)\, e^{A(t)},$$
$$K(t, s) = k(t, s)\, e^{A(t) - A(s) - A(t-s)}.$$

Therefore, it is easy to find that the existence and uniqueness properties of Eq. (1.1) are equivalent to those of the following equation (for the sake of simplicity, we still use the notation of $u(t)$, $f(t)$ and $k(t, s)$),

$$u'(t) + \int_0^t k(t, s)\, u(t-s)\, u(s)\, ds = f(t), \qquad 0 \leqslant t \leqslant T,\ u(0) = c, \tag{2.2}$$

where the functions $f(t)$ and $k(t, s)$ are continuous for $0 \leqslant t \leqslant T$ and $0 \leqslant s \leqslant t \leqslant T$, respectively, and c is a constant.

Theorem 2.1. *Equation* (2.2) *possesses a unique solution* $u \in C^1(\bar{I})$, *where* $\bar{I} = [0, T]$. (*Here,* $C^n(J)$ *denotes the space of functions with continuous nth derivatives* *on the interval J when n is a positive integer, and $C(J)$ denotes the space of continuous functions on the interval J.)*

Proof. *Step* 1: *Construct an operator S.* Let $V = C(\bar{I})$. Choosing an arbitrary function $v \in V$, one defines $u = S(v)$ to be the solution of

$$u'(t) + \int_0^t k(t, s)\, v(t-s)\, v(s)\, ds = f(t), \qquad 0 \leqslant t \leqslant T,\ u(0) = c, \tag{2.3}$$

that is,

$$u(t) = c - \int_0^t \int_0^r k(r, s)\, v(r-s)\, v(s)\, ds\, dr + \int_0^t f(s)\, ds. \tag{2.4}$$

Step 2: *S: $V \to V$ is a compact operator.* From (2.4), if $v \in V$, then $S(v) = u \in V$. Let X be a bounded subset of V, namely $\|v\| := \max_{0 \leqslant t \leqslant T} |v(t)|$ is less than a constant for any $v \in X$. From (2.3), there exists a constant M, such that $\|u'\| \leqslant M$ for any $v \in X$. By the mean-value theorem, one has $|u(x) - u(y)| \leqslant M\,|x - y|$ for all x, $y \in \bar{I}$. In particular, if one chooses $y = 0$, then $|u(x)| \leqslant |c| + MT$ for all $x \in \bar{I}$. Hence, by the Ascoli–Arzela theorem, there exists a convergent subsequence of the set $\{u = S(v) \mid v \in X\}$. S: $V \to V$ is thereby a compact operator.

Step 3: *Conclusion.* Let $Q = \{v \in V \mid |v(t)| \leqslant he^{rt},\ 0 \leqslant t \leqslant T\}$, where $r = 4KT(FT + |c|)$, $h = 2(FT + |c|)$, $K = \max_{0 \leqslant s \leqslant t \leqslant T} |k(t, s)|$, $F = \max_{0 \leqslant t \leqslant T} |\,(t)|$. It is readily verified that Q is a closed convex and bounded set in V. For any $v \in Q$, one has

$$\begin{aligned}
|u'(t)| &= \left| -\int_0^t k(t, s)\, v(t-s)\, v(s)\, ds + f(t) \right| \\
&\leqslant \int_0^t |k(t, s)|\, |v(t-s)|\, |v(s)|\, ds + F \\
&\leqslant \int_0^t Kh^2\, e^{r(t-s)}\, e^{rs}\, ds + F \\
&\leqslant Kh^2\, e^{rt} T + F.
\end{aligned}$$

Since inequality (2.5) is valid for $0 \leqslant t \leqslant T$, one has

$$\begin{aligned}
|u(t)| &= \left| \int_0^t u'(s)\, ds + c \right| \\
&\leqslant \int_0^t |u'(s)|\, ds + |c| \\
&\leqslant \int_0^t (Kh^2\, e^{rs} T + F)\, ds + |c|
\end{aligned}$$

$$
\begin{aligned}
&= Kh^2T\frac{1}{r}(e^{rt}-1)+Ft+|c| \\
&\leqslant Kh^2T\frac{1}{r}e^{rt}+FT+|c| \\
&= (FT+|c|)(e^{rt}+1) \\
&\leqslant he^{rt}
\end{aligned}
$$

and therefore, S maps Q into itself. S being a compact operator has a fixed point $u \in Q$ by Schauder's fixed point theorem. One can easily show that u is a solution of (2.2). Since $u \in Q$, and from (2.4), we know that $u \in C^1(\bar{I})$.

If $u, v \in C^1(\bar{I})$ are the two solutions of Eq. (2.2), one can obtain

$$
e'(t) + \int_0^t w(t, s)\, e(s)\, ds = 0, \qquad 0 \leqslant t \leqslant T,\ e(0) = 0, \tag{2.6}
$$

where $e(t) = u(t) - v(t)$ and $w(t, s) = k(t, t-s)\, u(t-s) + k(t, s)\, v(t-s)$.

Since $u(t)$, $v(t)$ and $k(t, s)$ are continuous functions, there exists a positive constant B such that $\max_{0 \leqslant t \leqslant T} \int_0^t w^2(t, s)\, ds \leqslant B$. From (2.6), one can get

$$
\begin{aligned}
(e^2(t))' &= -2e(t) \int_0^t w(t, s)\, e(s)\, ds \\
&\leqslant e^2(t) + \left(\int_0^t w(t, s)\, e(s)\, ds \right)^2 \\
&\leqslant e^2(t) + \int_0^t w^2(t, s)\, ds \int_0^t e^2(s)\, ds \\
&\leqslant e^2(t) + B \int_0^t e^2(s)\, ds.
\end{aligned}
$$

Let $Z(t) = \int_0^t e^2(s)\, ds$; then one has

$$
Z''(t) \leqslant Z'(t) + BZ(t), \qquad 0 \leqslant t \leqslant T;\ Z(0) = Z'(0) = 0.
$$

Furthermore, one can verify that

$$
(e^{-pt}(Z(t)\, e^{-qt})')' \leqslant 0, \qquad 0 \leqslant t \leqslant T;\ Z(0) = Z'(0) = 0, \tag{2.7}
$$

where $p = -(1+4B)^{1/2}$ and $q = \frac{1}{2}(1-p)$. Since $Z(t) \geqslant 0$ for $0 \leqslant t \leqslant T$, one can get $Z(t) = 0$ for $0 \leqslant t \leqslant T$ from (2.7). Hence $e(t) = 0$ for $0 \leqslant t \leqslant T$; i.e., Eq. (2.2) has a unique solution in the space $C^1(\bar{I})$. The proof of Theorem 2.1 is thereby complete.

COROLLARY 2.1. *Equation* (1.1) *possesses a unique solution* u *in the space* $C^1(\bar{I})$.

THEOREM 2.2. *If the functions* $a(t), f(t) \in C^n(\bar{I})$, $\partial^{(j)} k(t, s)/\partial t^{j-i} \partial s^i$, $0 \leqslant i \leqslant j \leqslant n$, *are continuous for* $0 \leqslant s \leqslant t \leqslant T$, *then Eq.* (1.1) *possesses a unique solution* $u \in C^{n+1}(\bar{I})$.

Proof. By Corollary 2.1, there exists a unique solution $u \in C^1(\bar{I})$, such that

$$u'(t) = -a(t)\, u(t) - \int_0^t k(t, s)\, u(t-s)\, u(s)\, ds + f(t), \qquad 0 \leqslant t \leqslant T, u(0) = c. \tag{2.8}$$

Under the assumptions of the theorem and by (2.8), one can easily show that $u \in C^{n+1}(\bar{I})$. The proof of the theorem is therefore complete.

We now know that the smoothness properties of the solution of Eq. (1.1) depend merely on those of the functions $a(t)$, $f(t)$ and $k(t, s)$. Thus we can use higher-order numerical methods to solve Eq. (1.1) when those functions are smooth.

3. NUMERICAL METHODS

In this section, we introduce implicit Runge–Kutta methods with m stages and optimal order $p = 2m$ for finding the numerical solution of (1.1). These methods were originally used for Volterra integro-differential equations (see Brunner [5]). We use similar ideas to determine the approximate solution of (1.1). The underlying theoretical results for these methods are discussed in [4]. The results of the numerical solution of several examples are summarized in the next section.

Implicit Runge–Kutta methods can be viewed as fully discretized collocation methods in certain polynomial spline spaces. The polynomial spline space used for the approximation of the exact solution of Eq. (1.1) is defined as follows: let $N \geqslant 1$, $m \geqslant 1$ (with N, m are positive integers), $0 = t_0 < t_1 < \cdots < t_N = T$, $Z_N = \{t_n : n = 0, ..., N-1\}$, $\bar{Z}_N = Z_N \cup T$, and set $r_n = [t_n, t_{n+1}]$ $(n = 0, ..., N-1)$. Then

$$S^{(0)}(Z_N) = \{u \in C(I) : u \,|\, r_n = u_n \in P_m \ (n = 0, ..., N-1)\}, \tag{3.1}$$

with $\dim(S_m^{(0)}(Z_N)) = Nm + 1$, is our approximating polynomial spline space, where $P_m = \{p(x) \,|\, p(x)$ is a polynomial function, degree of $p(x) \leqslant m\}$. Let $X(N) = \bigcup_{n=0}^{N-1} X_n$, with

$$X_n = \{t_n + c_i h \,|\, 0 \leqslant c_1 < \cdots < c_m \leqslant 1\} \qquad (n = 0, ..., N-1) \tag{3.2}$$

denote the set of collocation points at which the desired approximation $y \in S_m^{(0)}(Z_N)$ is to satisfy the given integro-differential equation (1.1). This approximation is thus determined recursively by

$$y_n'(t_n+c_ih)+a(t_n+c_ih)\,y_n(t_n+c_ih)+\int_{t_n}^{t_n+c_ih} k(t_n+c_ih, S)\,y_n(S)\,y_0(t_n+c_ih-s)\,ds$$
$$+\sum_{k=0}^{n-1}\int_{t_k}^{t_k+c_ih} k(t_n+c_ih, s)\,y_k(s)\,y_{n-k}(t_n+c_ih-s)\,ds$$
$$+\sum_{k=0}^{n-1}\int_{t_k+C_ih}^{t_{k+1}} k(t_n+c_ih, s)\,y_k(s)\,y_{n-k-1}(t_n+c_ih-s)\,ds=f(t_n+c_ih),$$
$$i=1, ..., m\ (n=0, ..., N-1). \quad (3.3)$$

For $s\in[t_k, t_{k+1}]$, let $s=t_k+\theta h$, $\theta\in[0,1]$, $k=0, ..., N-1$; then

$$y_n'(t_n+c_i\,h)+a(t_n+c_ih)\,y_n(t_n+c_ih)+h\int_0^{c_i} k(t_n+c_ih, t_n+\theta h)$$
$$\times y_n(t_n+\theta h)\,y_0((c_i-\theta)\,h)\,d\theta+h\sum_{k=0}^{n-1}\int_0^{c_i} k(t_n+c_ih, t_k+\theta h)$$
$$\times y_k(t_k+\theta h)\,y_{n-k}(t_{n-k}+(c_i-\theta)\,h)\,d\theta+h\sum_{k=0}^{n-1}\int_{c_i}^{1} k(t_n+c_ih, t_k+\theta h)\,y_k+\theta h)$$
$$\times y_{n-k-1}(t_{n-k-1}+(1+c_i-\theta)\,h)\,d\theta=f(t_n+c_ih),$$
$$i=1, ..., m, (n=0, ..., N-1). \quad (3.4)$$

Since $y\in C(\bar{I})$, we have

$$y_n(t_n)=y_{n-1}(t_n), \qquad t_n\in Z_N \text{ (with } y_0(0)=c). \quad (3.5)$$

In most cases the integrals occurring in the collocation equation (3.5) cannot be found analytically but have to be approximated by suitable quadrature formulas (compare also [5]). This means that, instead of $y\in S_m^{(0)}(Z_N)$, we compute an approximation $\hat{y}\in S_m^{(0)}(Z_N)$ from a perturbed collocation equation,

$$\hat{y}_n'(t_n+c_ih)+a(t_n+c_ih)\,\hat{y}_n(t_n+c_ih-+h\sum_{j=1}^{m} w_{ij}k(t_n+c_ih, t_n+c_ic_jh)$$
$$\times \hat{y}_n(t_n+c_ic_jh)\,\hat{y}_0((c_i-c_ic_j)\,h)+h\sum_{k=0}^{n-1}\sum_{j=1}^{i} w_jk(t_n+c_ih, t_k+c_jh)$$
$$\times \hat{y}_k(t_k+c_jh)\,\hat{y}_{n-k}(t_{n-k}+(c_i-c_j)\,h)+h\sum_{k=0}^{n-1}\sum_{j=i+1}^{m} w_jk(t_n+c_ih, t_k+c_jh)$$
$$\times \hat{y}_k(t_k+c_jh)\,\hat{y}_{n-k-1}(t_{n-k-1}+(1+c_i-c_j)\,h)=f(t_n+c_ih),$$
$$i=1, ..., m\ \ (n=0, ..., N-1), \quad (3.6)$$

where we use the m-point Gauss quadrature formulas.

We have proved that

$$\|u-\hat{y}\|_\infty = \max_{0 \leqslant t \leqslant T} |u(t)-y(t)| = \mathcal{O}(h^m) \tag{3.7}$$

$$\|u'-\hat{y}'\|_\infty = \max_{0 \leqslant t \leqslant T} |u'(t)-y'(t)| = \mathcal{O}(h^m) \tag{3.8}$$

and

$$|u(t_n)-y(t_n)| = \mathcal{O}(h^{2m}), \qquad t_n \in Z_N \tag{3.9}$$

(see [4]).

Now we rewrite the discretized collocation equation (3.6) in a form which exhibits more clearly the fact that (3.6) defines a class of implicit Runge–Kutta methods for the solution of Eq. (1.1). Define $Y_j^{(n)} = \hat{y}_n'(t_n + c_j h)$ (where $\hat{y}_n' \in P_{m-1}$) and set

$$q_j(x) = \prod_{\substack{r=1 \\ r \neq j}} \frac{x-c_r}{c_j-c_r} \qquad (j=1, ..., m). \tag{3.10}$$

Since $\hat{y}_n'(t_n+\theta h) = \sum_{j=1}^m q_j(\theta)\, Y_j^{(n)}$, we have

$$\hat{y}_n(t_n+\theta h) = \hat{y}_n + h \sum_{j=1}^m b_j(\theta)\, Y_j^{(n)}, \tag{3.11}$$

where we have set $\hat{y}_n = \hat{y}_n(t_n)$ $(=\hat{y}_{n-1}(t_n))$, and

$$b_j(x) = \int_0^x q_j(u)\, du, \qquad j=1,..., m. \tag{3.12}$$

Thus, the fully discretized collocation equation can be written in the form

$$\begin{aligned}
&Y_i^{(n)} + a(t_n+c_i h)\left(\hat{y}_n + h\sum_{j=1}^m b_j(c_i)\, Y_j^{(n)}\right) + h\sum_{r=1}^m w_{ir} k(t_n+c_i h,\, t_n + c_i c_r h)\\
&\quad \times\left(\hat{y}_n + h\sum_{j=1}^m b_j(c_r c_i)\, Y_j^{(n)}\right)\left(\hat{y}_0 + h\sum_{j=1}^m b_j(c_i - c_i c_r)\, Y_j^{(0)}\right)\\
&\quad + h\sum_{k=0}^{n-1}\sum_{r=1}^{i} w_r k(t_n + c_i h,\, t_k + c_r h)\left(\hat{y}_k + h\sum_{j=1}^m b_j(c_r)\, Y_j^{(k)}\right)\\
&\quad \times\left(\hat{y}_{n-k} + h\sum_{j=1}^m b_j(c_i - c_r)\, Y_j^{(n-k)}\right)\\
&\quad + h\sum_{k=0}^{n-1}\sum_{r=i+1}^{m} w_r k(t_n + c_i h,\, t_k + c_r h)\left(\hat{y}_k + h\sum_{j=1}^m b_j(c_r)\, Y_k^{(k)}\right)\\
&\quad \times\left(\hat{y}_{n-k-1} + n\sum_{j=1}^m b_j(1 + c_i - c_r)\, Y_j^{(n-k-1)}\right) = f(t_n + c_i h),
\end{aligned}$$

$$i=1, ..., m \ (n=0, ..., N-1). \tag{3.13}$$

Rearranging the relation (3.13), we obtain for $n=0$,

$$Y_i^{(0)}+\sum_{j=1}^{m}\left[ha(c_ih)b_j(c_i)+\hat{y}_0h^2\left(\sum_{r=1}^{m}w_{ir}k(c_ih, c_ic_rh)(b_j(c_ic_r)+b_j(c_i-c_ic_r))\right)\right]$$
$$\times Y_j^{(0)}+h^3\sum_{k=1}^{m}\sum_{j=1}^{m}\left[\sum_{r=1}^{m}w_{ir}k(c_ih, c_ic_rh)\,b_j(c_ic_r)\,b_k(c_i-c_ic_r)\right]$$
$$\times Y_j^{(0)}Y_k^{(0)}+\hat{y}_0a(c_ih)+h\hat{y}_0^2\sum_{j=1}^{m}w_{ij}k(c_ih, c_ic_jh)=f(c_ih), \qquad i=1, ..., m, \quad (3.14)$$

where $\hat{y}_0=c$, and for $n\geqslant 1$,

$$Y_i^{(n)}+\sum_{j=1}^{m}A_{ij}^{(n)}Y_j^{(n)}=-B_i^{(n)}\hat{y}_n+C_i^{(n)}, \qquad i=1, ..., m\ (n=1, ..., N-1), \qquad (3.15a)$$

where

$$A_{ij}^{(n)}=ha(t_n+c_ih)\,b_j(c_i)+h^2\sum_{r=1}^{m}w_{ir}k(t_n+c_ih, t_k+c_ic_rh)\,b_j(c_ic_r)$$
$$\times\left(\hat{y}_0+h\sum_{k=1}^{m}b_k(c_i(1-c_r))\,Y_k^{(0)}\right)+h^2\sum_{r=1}^{i}w_rk(t_n+c_ih, c_rh)$$
$$\times b_j(c_i-c_r)\left(\hat{y}_0+h\sum_{k=1}^{m}b_k(c_r)\,Y_k^{(0)}\right), \qquad (3.15b)$$

$$B_i^{(n)}=a(t_n+c_ih)+h\sum_{r=1}^{m}w_{ir}k(t_n+c_ih, t_n+c_ic_rh)$$
$$\times\left(\hat{y}_0+h\sum_{k=1}^{m}b_k(c_i(1-c_r))\,Y_k^{(0)}\right)$$
$$+h\sum_{r=1}^{i}w_rk(t_n+c_ih, c_rh)\left(\hat{y}_0+h\sum_{k=1}^{m}b_k(c_r)\,Y_k^{(0)}\right), \qquad (3.15c)$$

$$C_i^{(n)}=f(t_n+c_ih)-h\sum_{k=1}^{n-1}\sum_{r=1}^{i}w_rk(t_n+c_ih, t_k+c_rh)\left(\hat{y}_k+h\sum_{j=1}^{m}b_j(c_r)\,Y_j^{(k)}\right)$$
$$\times\left(\hat{y}_{n-k}+h\sum_{j=1}^{m}b_j(c_i-c_r)\,Y_j^{(n-k)}\right)$$
$$-h\sum_{k=0}^{n-1}\sum_{r=i+1}^{m}w_rk(t_n+c_ih, t_k+c_rh)$$
$$\times\left(\hat{y}_k+h\sum_{j=1}^{m}b_j(c_r)\,Y_j^{(k)}\right)\left(\hat{y}_{n-k-1}+h\sum_{j=1}^{m}b_j(1+c_i-c_r\ Y_j^{n-k-1})\right). \qquad (3.15d)$$

We also have

$$\hat{y}_n = \hat{y}_{n-1}(t_{n-1} + h) = \hat{y}_{n-1} + h \sum_{j=1}^{m} b_j(1)\, Y_j^{(n-1)} \qquad (n = 1, \ldots, N). \tag{3.16}$$

Note that (3.13) is a nonlinear system. Fortunately, we can compute $Y_i^{(n)}$ and y_n ($n \geqslant 1$, $i = 1, \ldots, m$) by (3.15) and (3.16) through solving some systems of linear equations if we know the initial value $Y_i^{(0)}$ ($i = 1, \ldots, m$). Newton iteration or even a simpler one can be employed to determine the initial value from (3.14). It can be justified that for any given starting value $Y_i^{(0),0}$ ($i = 1, \ldots, m$) the iteration is convergent whenever $h > 0$ is sufficiently small.

4. Numerical Experiments

In Section 3, we have seen that an implicit Runge–Kutta method (3.13) or (3.14)–(3.16) for (1.1) is characterized by the following arrays: the collocation parameters $\{c_i; i = 1, \ldots, m\}$, the quadrature weights of Gauss quadrature formulas $\{w_{ij}; i, j = 1, \ldots, m\}$ and $\{w_j; j = 1, \ldots, m\}$. We choose them appropriately so that the results (3.7), (3.8) and (3.9) are valid (see [4] or [5]): $m = 2$; $c_1 = (3 - \sqrt{3})/6$, $c_2 = (3 + \sqrt{3})/6$; $w_1 = w_2 = \frac{1}{2}$; and

$$W = (w_{ij}) = \begin{pmatrix} (3 - \sqrt{3})/12 & (3 - \sqrt{3})/12 \\ (3 + \sqrt{3})/12 & (3 + \sqrt{3})/12 \end{pmatrix}$$

$(b_1(x); b_2(x)) = (x(x - 2c_2)/2(c_1 - c_2);\ x(x - 2c_1)/2(c_2 - c_1))$.

We consider the following examples:

Example 1.

$$u'(t) + \frac{1}{8} e^{-2t} u(t) + \int_0^t \frac{1}{2} e^{-(t+s)} u(t-s)\, u(s)\, ds$$
$$= -\frac{1}{4} e^{-t} + \frac{1}{32} e^{-2t}, \qquad 0 \leqslant t \leqslant 4,$$
$$u(0) = \frac{1}{4}.$$

The exact solution is $u(t) = \frac{1}{4} e^{-t}$.

EXAMPLE 2.

$$u'(t)-\frac{2}{t+1}u(t)+\int_0^t \frac{1}{(t+1)^2(s+1)^2}u(t-s)\,u(s)\,ds$$
$$=\frac{1}{3}\left[t+1-\frac{1}{(t+1)^2}\right], \qquad 0\leqslant t\leqslant 4,$$
$$u(0)=1.$$

The exact solution is $u(t)=(t+1)^2$.

EXAMPLE 3.

$$u'(t)+u(t)-\frac{3}{5}\int_0^t u(t-s)\,u(s)\,ds=30(1+t)\cos 3t \qquad 0\leqslant t\leqslant 4,$$
$$u(0)=0.$$

The exact solution is $u(t)=10\sin 3t$. There are some difficulties in getting an accurate approximate solution for this example, because the derivative of the exact solution changes rapidly on the interval $[0, 4]$.

EXAMPLE 4.

$$u'(t)+u(t)+\int_0^t tsu(t-s)\,u(s)\,ds=\frac{t^5}{60}(t^2-10t+20)+t^2-2, \qquad 0\leqslant t\leqslant 4,$$
$$u(0)=0.$$

The exact solution is $u(t)=t(t-2)$.

We list in Tables I, II, III, and IV the resulting errors. By error we mean

$$\text{error}=|\text{exact value}-\text{approximate value}|.$$

The programs are written in FORTRAN in double precision for the Honeywell

TABLE I

Errors for Example 1

t	$h=0.1$	$h=0.05$	$h=0.025$
0.5	4.04×10^{-9}	2.19×10^{-10}	3.97×10^{-12}
1.0	4.18×10^{-9}	2.12×10^{-10}	7.45×10^{-12}
2.0	3.31×10^{-9}	1.64×10^{-10}	1.17×10^{-11}
3.0	2.81×10^{-9}	1.34×10^{-10}	1.36×10^{-11}
4.0	2.64×10^{-9}	1.23×10^{-10}	1.43×10^{-11}

TABLE II

Errors for Example 2

t	$h=0.1$	$h=0.05$	$h=0.025$
0.5	4.19×10^{-9}	2.04×10^{-9}	1.01×10^{-9}
1.0	6.91×10^{-9}	3.36×10^{-9}	1.66×10^{-9}
2.0	9.81×10^{-9}	4.79×10^{-9}	2.36×10^{-9}
3.0	1.08×10^{-9}	5.23×10^{-9}	2.57×10^{-9}
4.0	1.04×10^{-8}	5.02×10^{-9}	2.47×10^{-9}

TABLE III

Errors for Example 3

t	$h=0.1$	$h=0.05$	$h=0.025$
0.5	3.99×10^{-5}	2.36×10^{-6}	1.44×10^{-7}
1.0	6.85×10^{-5}	4.31×10^{-6}	2.69×10^{-7}
2.0	7.24×10^{-4}	4.54×10^{-5}	2.84×10^{-6}
3.0	4.29×10^{-3}	2.68×10^{-4}	1.68×10^{-5}
4.0	2.84×10^{-2}	9.59×10^{-3}	1.12×10^{-4}

TABLE IV

Errors for Example 4

t	$h=0.1$	$h=0.05$	$h=0.025$
0.5	1.99×10^{-9}	1.84×10^{-10}	1.39×10^{-11}
1.0	4.31×10^{-8}	3.17×10^{-9}	2.20×10^{-10}
2.0	7.87×10^{-7}	5.32×10^{-8}	3.50×10^{-9}
3.0	6.74×10^{-6}	4.47×10^{-7}	2.87×10^{-8}
4.0	7.00×10^{-5}	4.62×10^{-6}	2.94×10^{-7}

DPS8 at Peking University. It appears that the implicit Runge–Kutta method we used has two major advantages: stability and accuracy. The major drawback is that the algorithm we used above is somewhat more complicated to use than the multistep method presented in [1].

Note. The programs for the implicit Runge–Kutta method can be obtained from the authors.

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